Mutual linearity of nonequilibrium network currents

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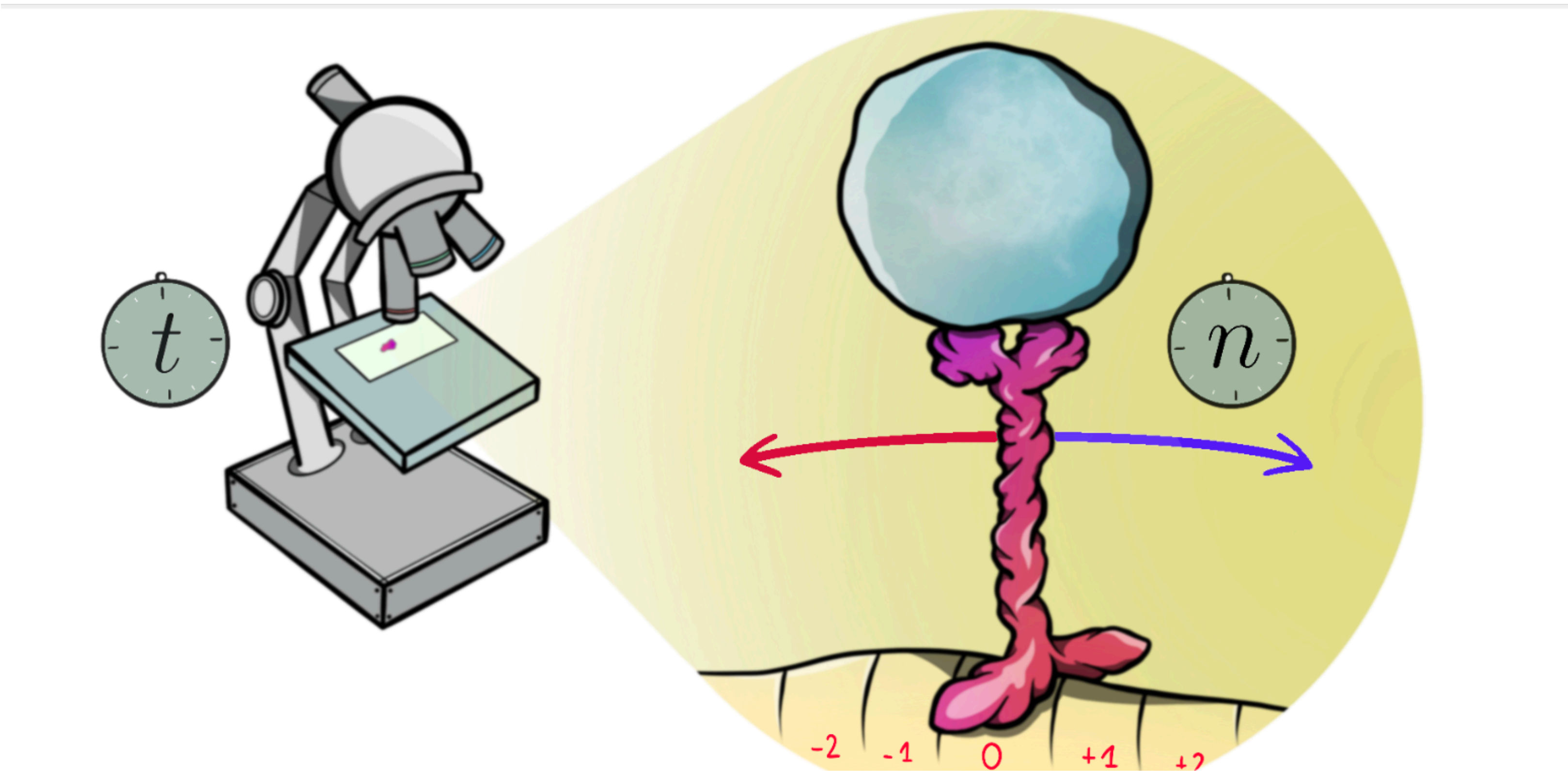
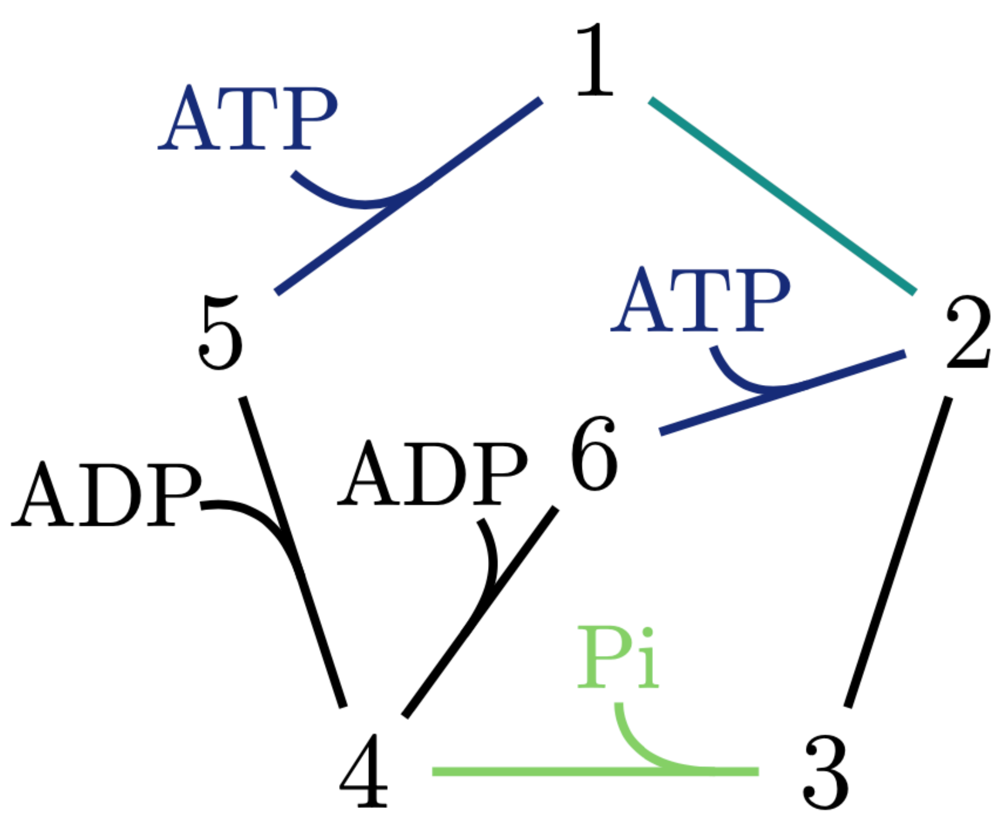
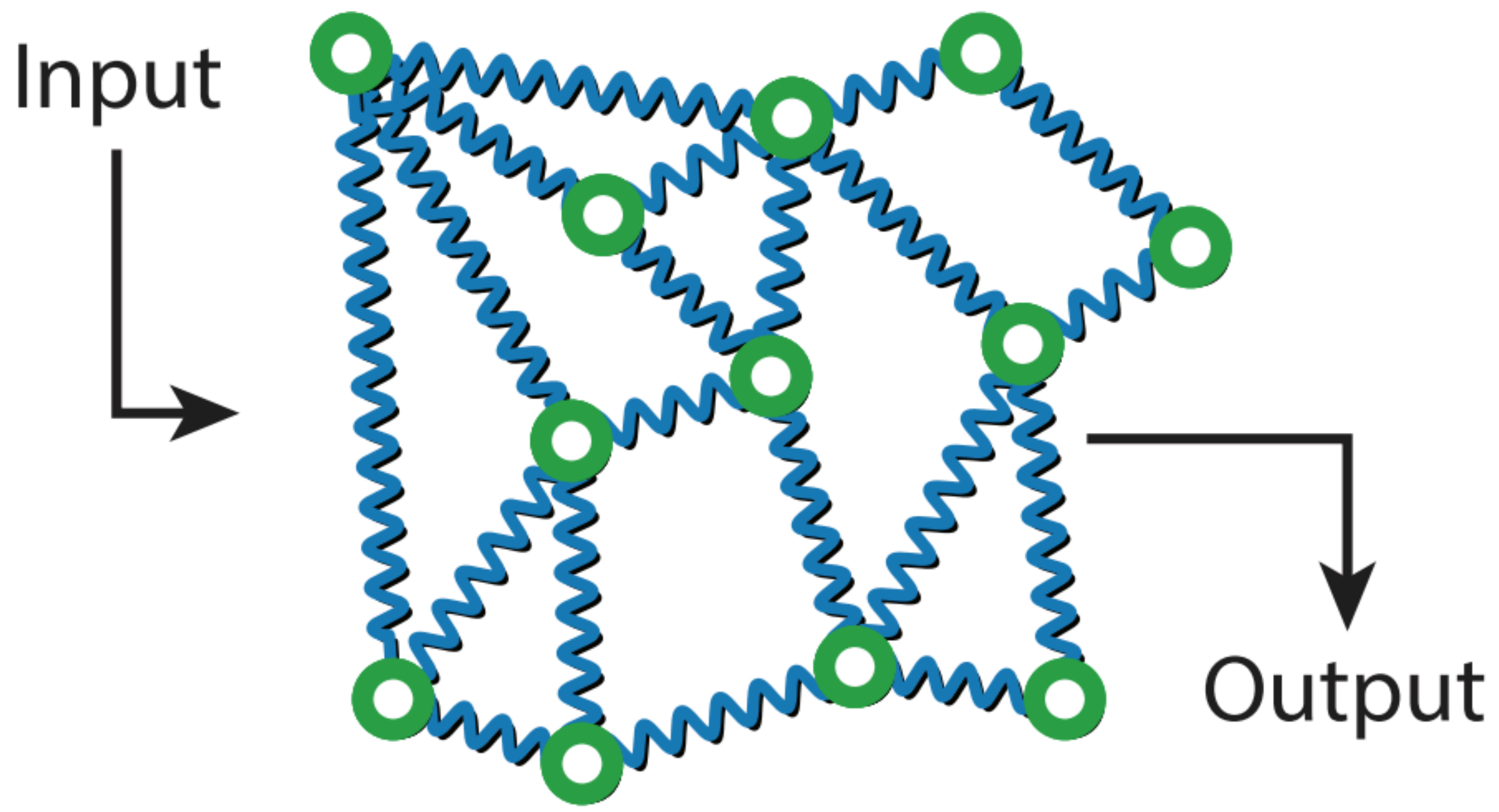
Networks in the real world

Ex. Chemical reaction networks, metabolic networks





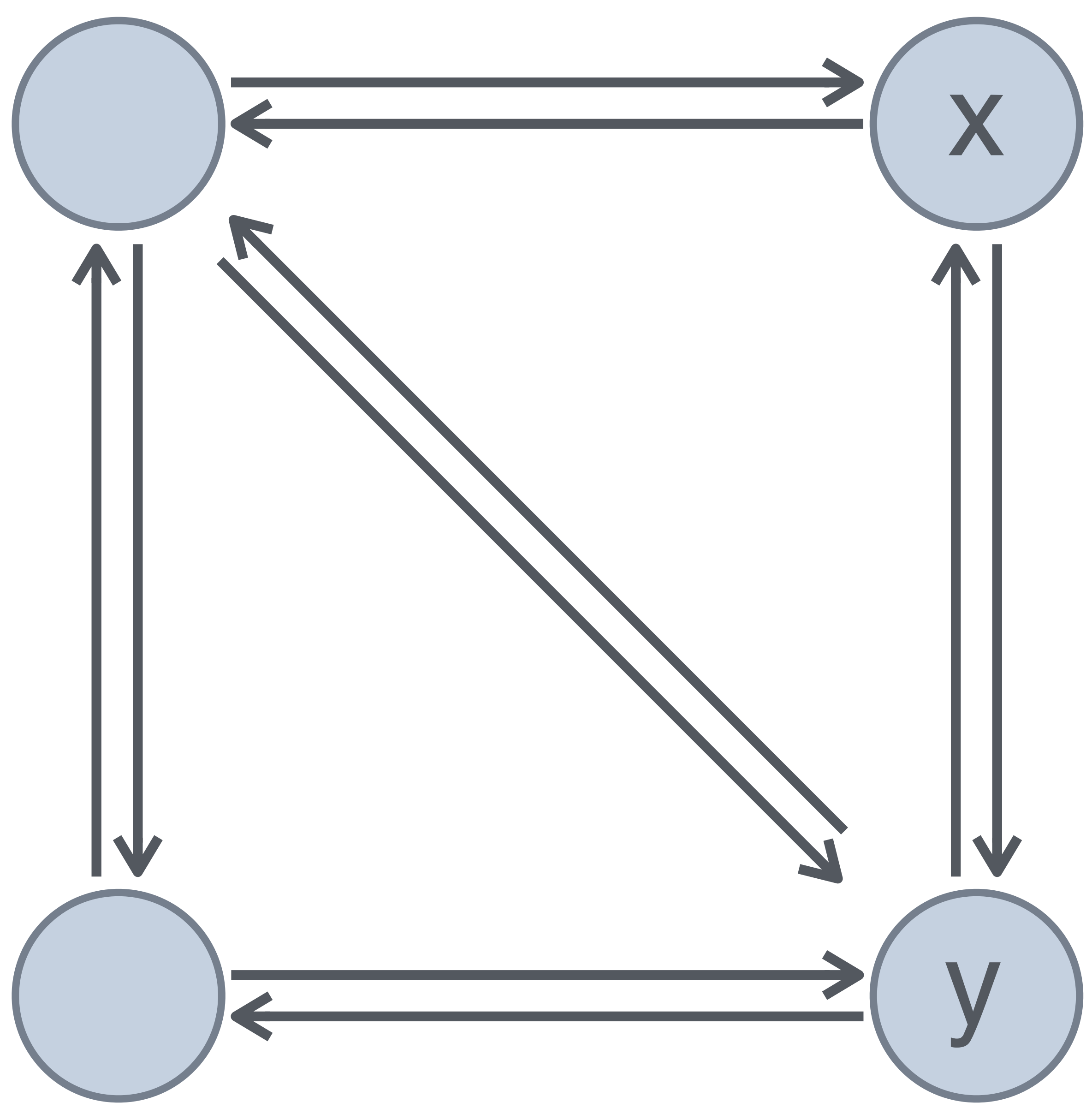
Ex. Pipe networks, resistor networks, elastic networks



[Stern and Murugan, Annu. Rev. Condens. Matter. Phys.14 (2023)]

Framework

Continuous-time Markov chain with rates $\boldsymbol{r} = \{r(\mathsf{x}\,|\,\mathsf{y})\}$ on a network (state space)



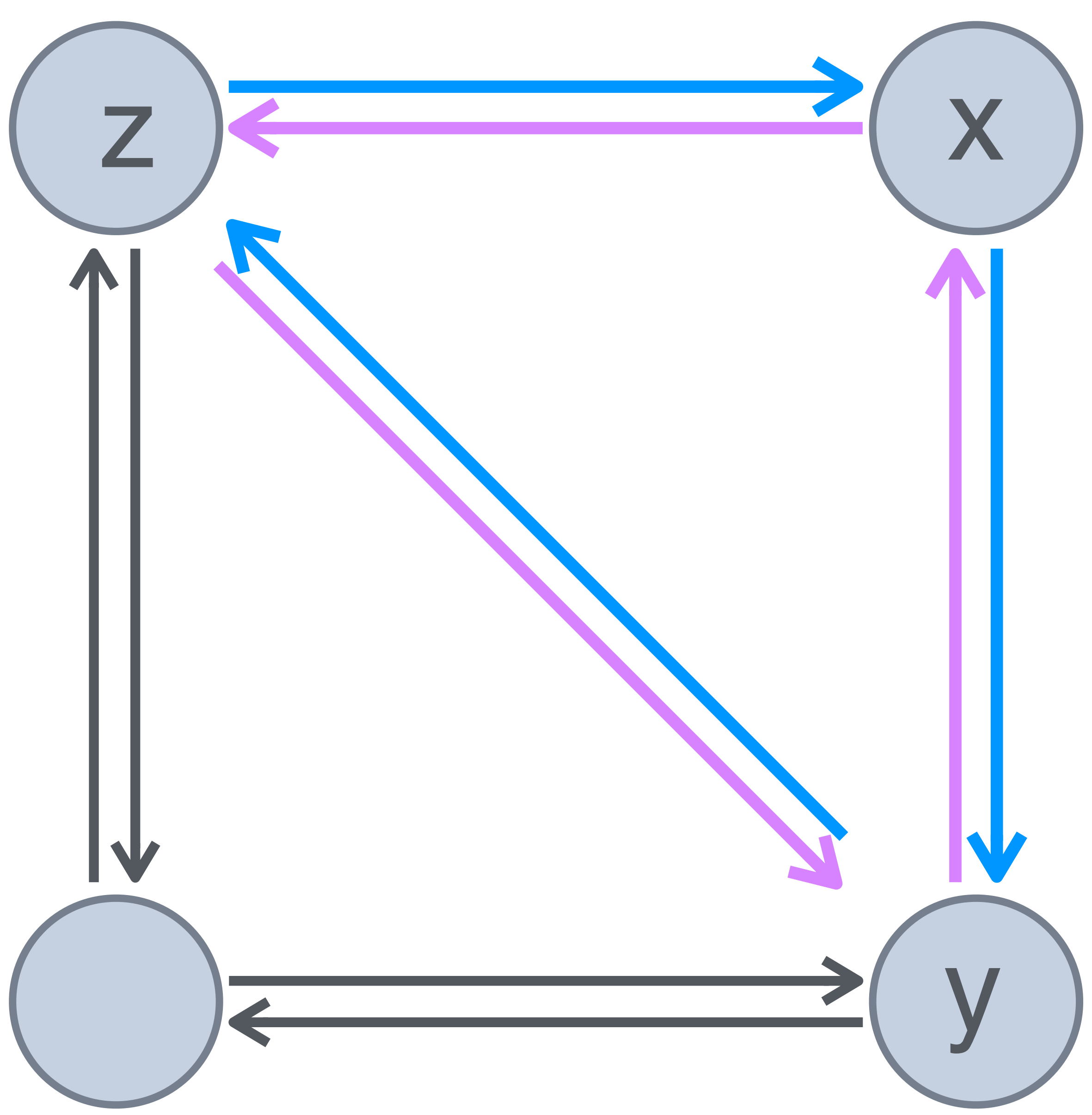
Cumulative edge current: $J_{\mathsf{x}\leftarrow\mathsf{y}}(t) = \#_{\mathsf{x}\leftarrow\mathsf{y}}(t) - \#_{\mathsf{y}\leftarrow\mathsf{x}}(t)$

Stationary edge current: $j_{\mathsf{xy}} = \lim_{t\to\infty} \dfrac{\langle J_{\mathsf{x}\leftarrow\mathsf{y}}(t)\rangle}{t}$

(1) We assume that the system **reaches a stationary state** (ensured by Perron-Frobenius theorem for connected network)

Framework

If the transition rates fulfill the **Kolmogorov condition**, the stationary state is **equilibrium** with vanishing stationary currents $j_{xy} = 0 \; \forall (x,y)$



$\forall$ cycle:

$$\boxed{r(z|x)r(y|z)r(x|y) = r(y|x)r(z|y)r(x|z)}$$

$\Leftrightarrow$ Detailed Balance (stat. mech.)

$\Leftrightarrow$ Wegscheider condition (chemistry)

$\Leftrightarrow$ Kirchhoff's voltage law (electric circuits)

Framework

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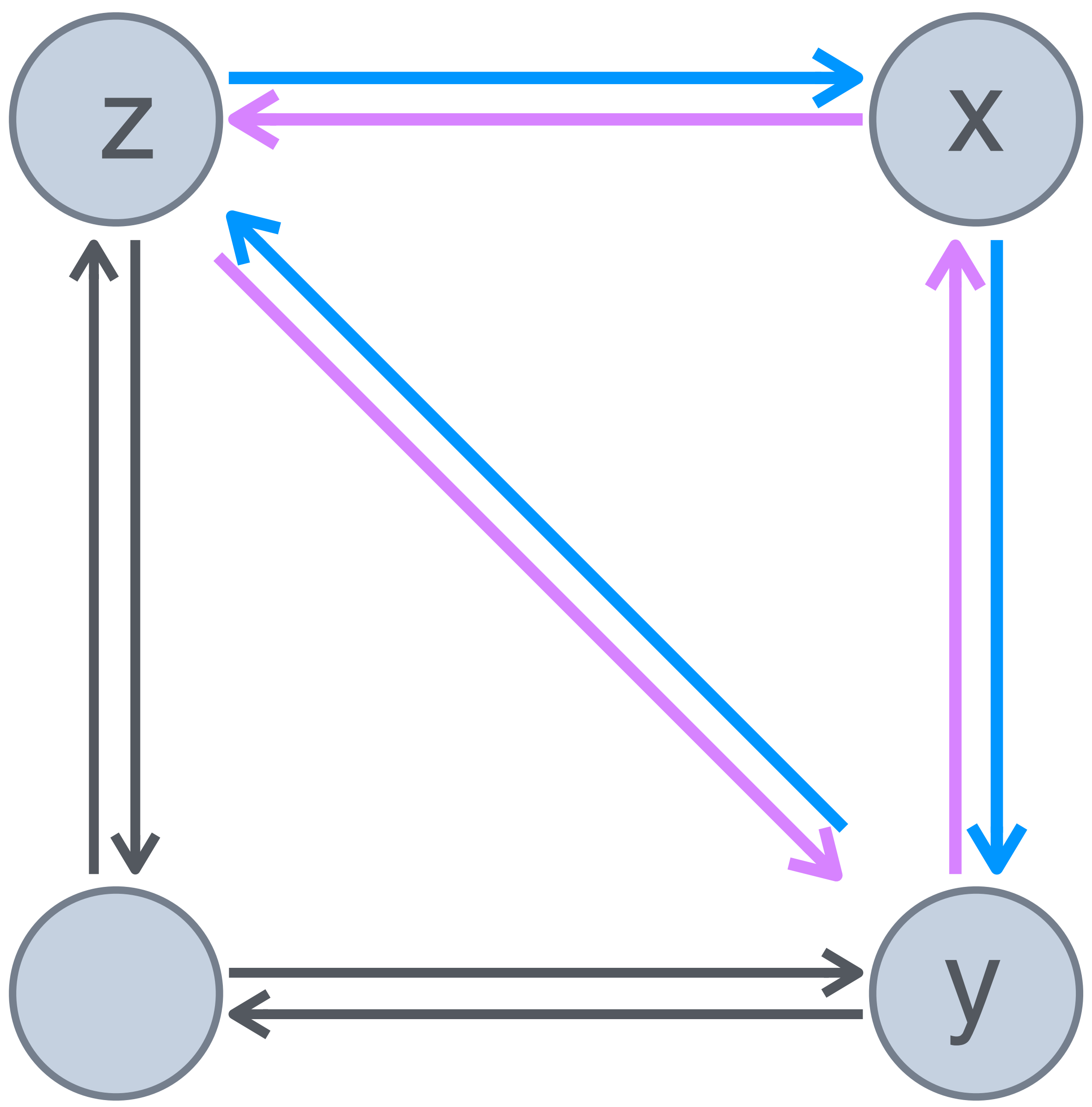


∀ cycle:

$$r(z|x)r(y|z)r(x|y) = r(y|x)r(z|y)r(x|z)$$

⇔ Detailed Balance (stat. mech.)

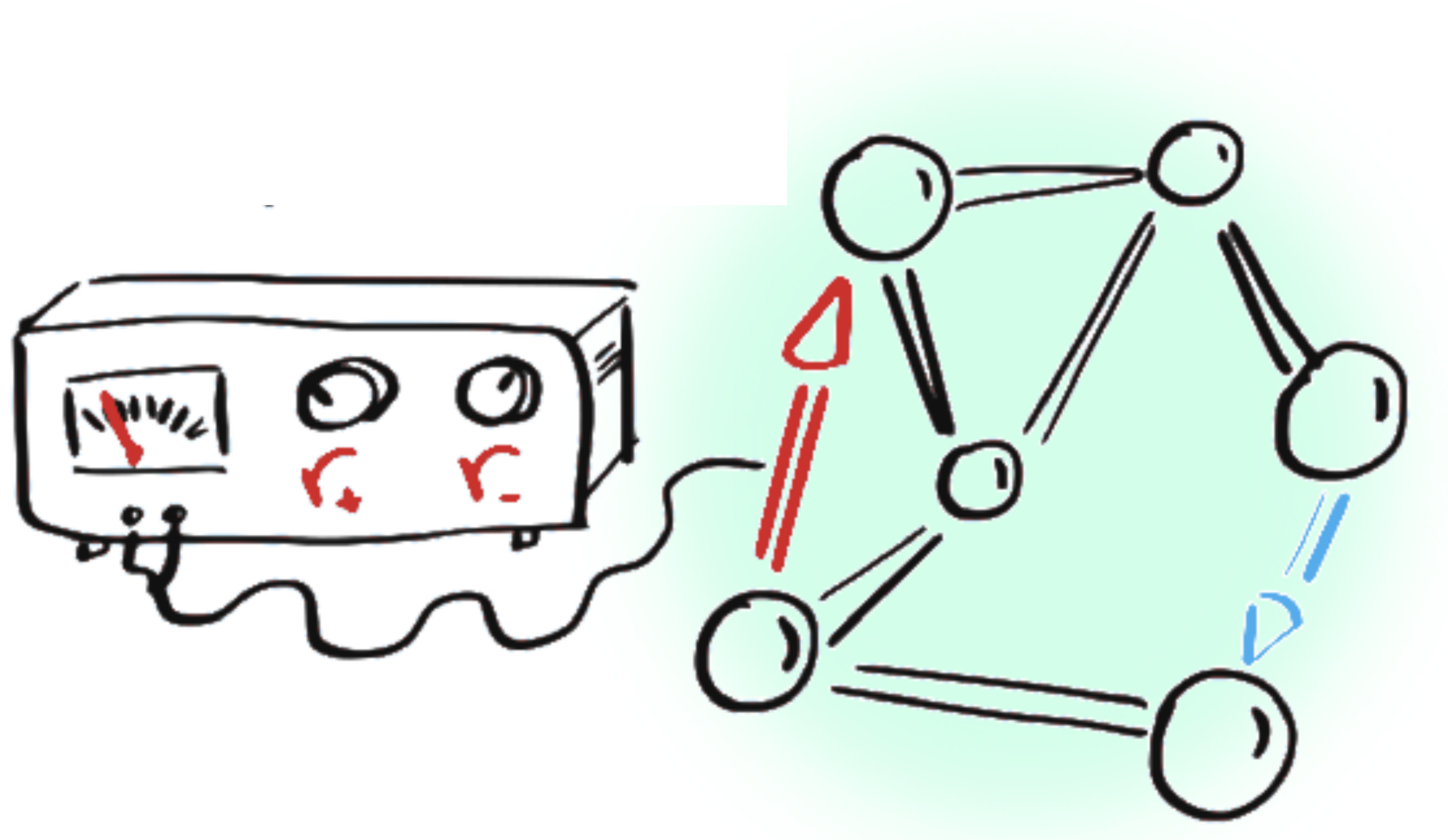
⇔ Wegscheider condition (chemistry)

⇔ Kirchhoff's voltage law (electric circuits)

(2) We take arbitrary transition rates ⇔ **far-from-equilibrium**

Questions

- How do currents respond to other currents in nonequilibrium steady-states?
- Can we establish **current-to-current relationships** when controlling the transition rates of one (or more) "input" current?

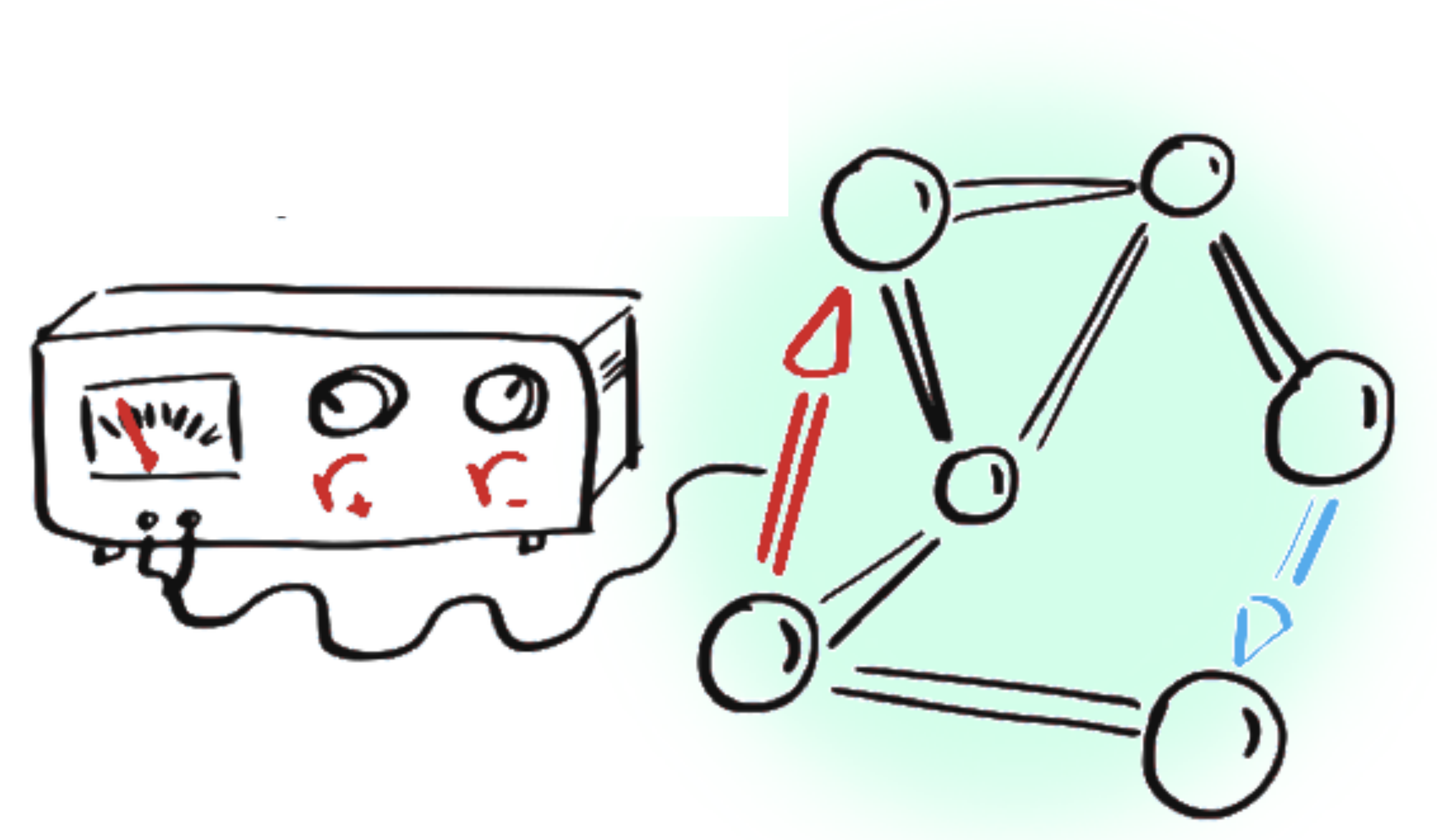


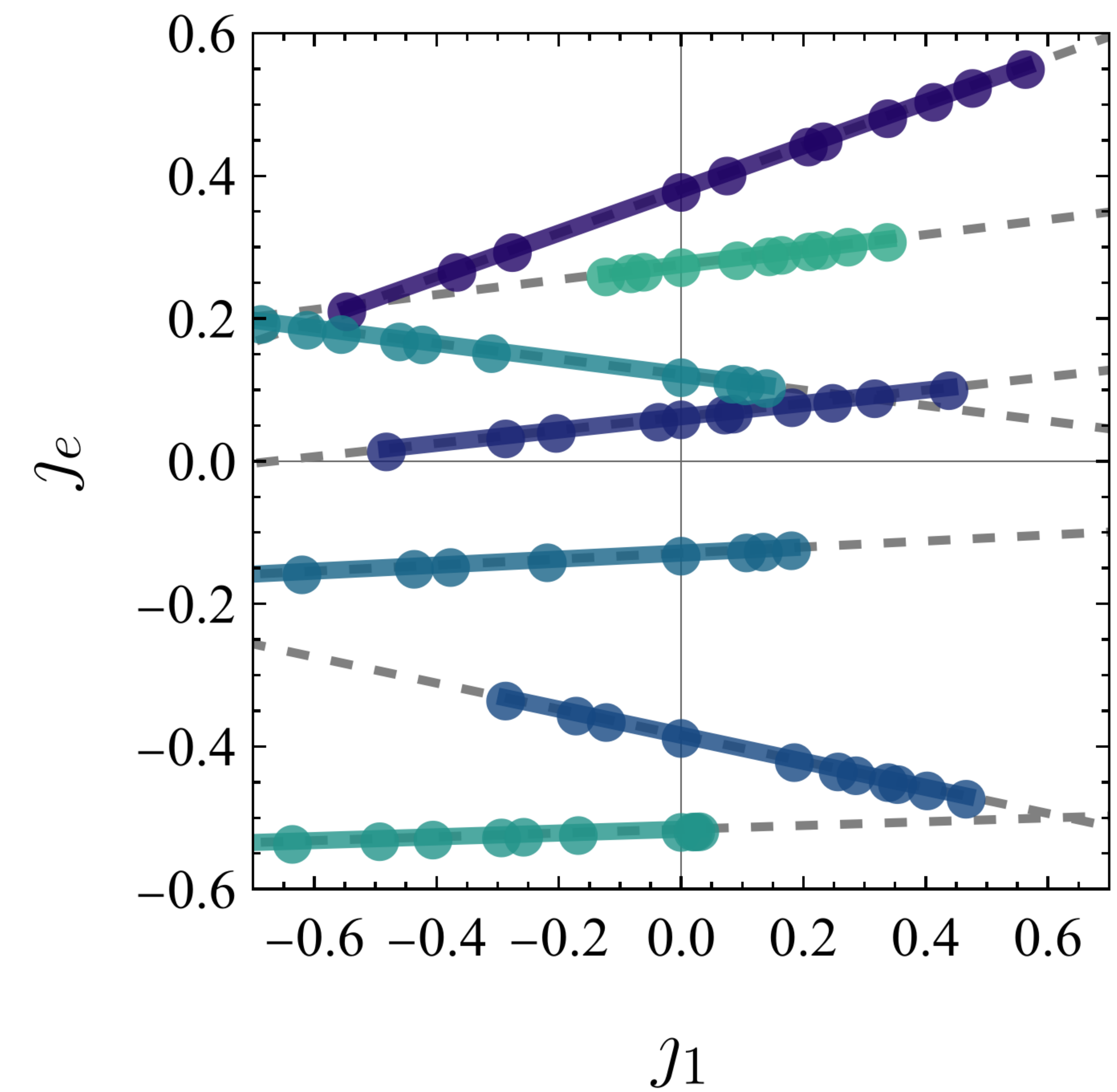
Ex. Metabolic recontruction…

The case of one input current

Main result: Control rates $r_{\pm i}$ and fix every other rate. Then $\forall j_{e \neq i}$ there exist two constants (independent on $r_{\pm i}$) such that there is a **linear-affine relation**

$$j_e(r_{\pm i}) = \lambda_e^0 + \lambda_e^1 \, j_i(r_{\pm i})$$





Path to the proofs

$$\boxed{j_e(r_{\pm i}) = \lambda_e^0 + \lambda_e^1 j_i(r_{\pm i})}$$

- The affine parameter λ_e^0 is the value of the **current "at stalling"**, when we set the input rates to zero

- The linear parameter λ_e^1 is the **current-to-current susceptibility**: $\lambda_e^1 = \partial_{r_{\pm i}} j_e / \partial_{r_{\pm i}} j_i$

The difficult part of the proof is to show that $\partial_{r_{\pm i}} j_e / \partial_{r_{\pm i}} j_i$ does not depend on the input rates…

Path to the proofs

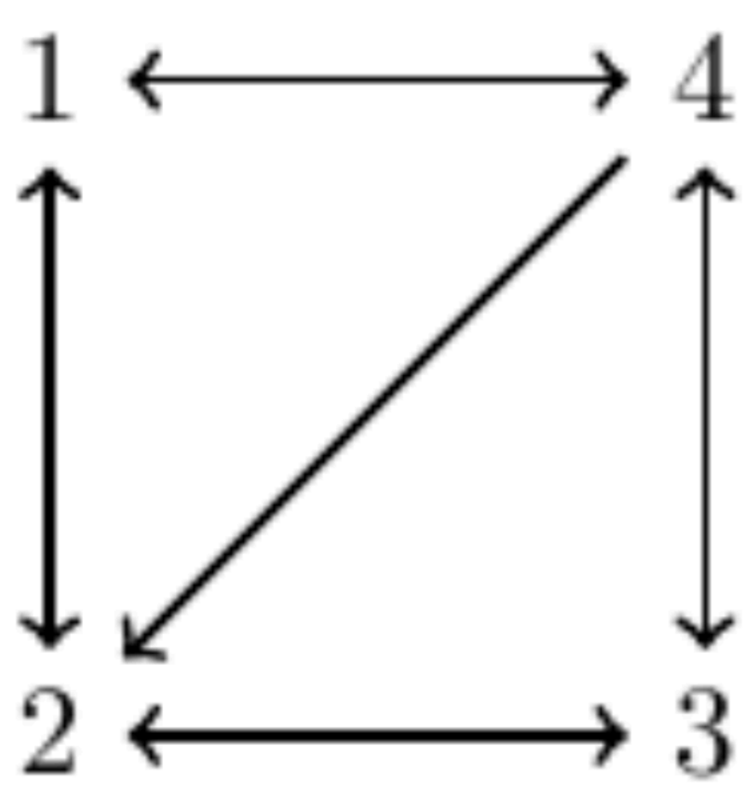
You can either…

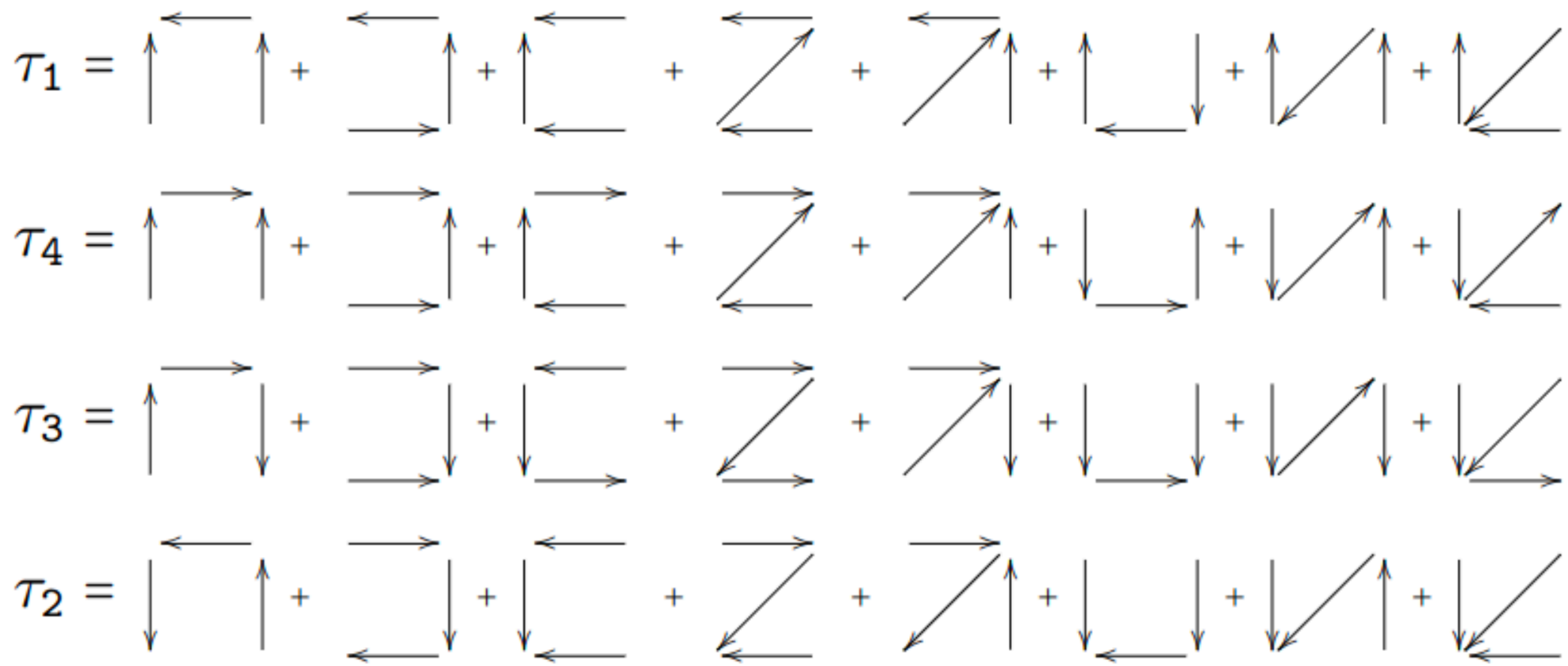
- Use graph theory
- Expand the rate matrix
- Use Laplace transform (go to frequency domain)

Spanning tree ensembles

- **Use graph theory**

Idea: Current in terms of **spanning trees** (Markov chain tree theorem)





Spanning tree ensembles

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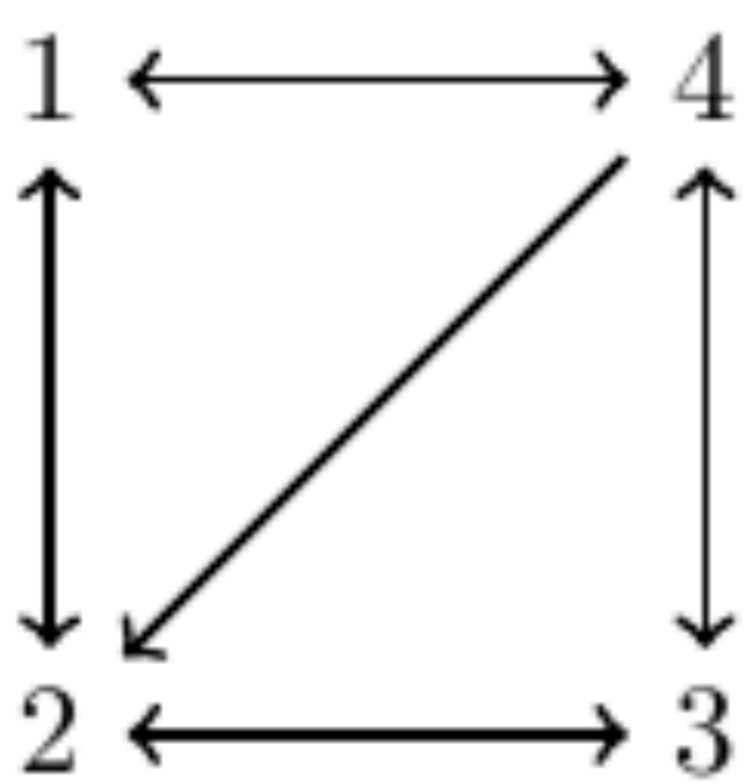

$$\boxed{j_{xy} = \frac{r_{xy}\tau_y - r_{yx}\tau_x}{\sum_z \tau_z}}$$

Currents are highly nonlinear in the rates!

Spanning tree ensembles

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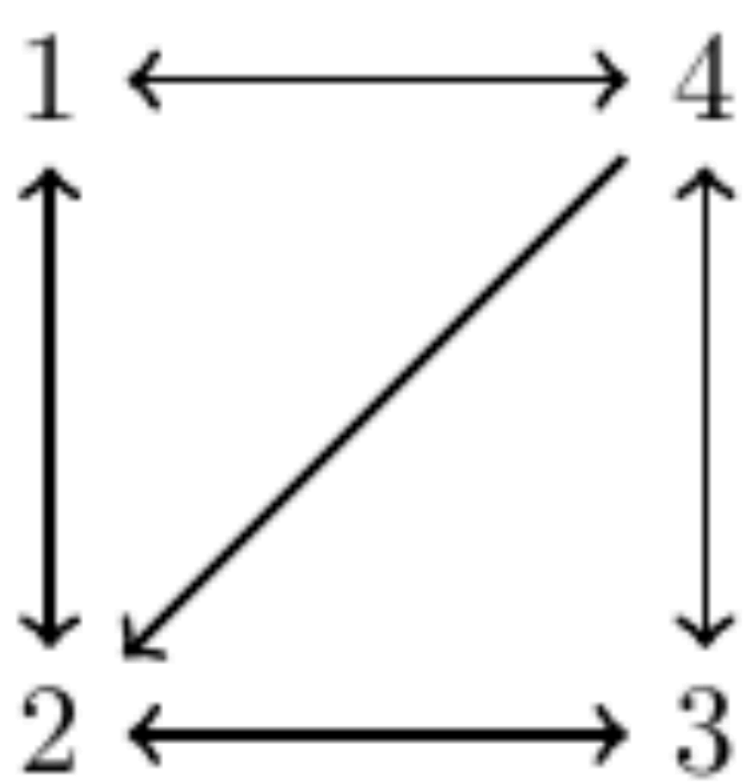

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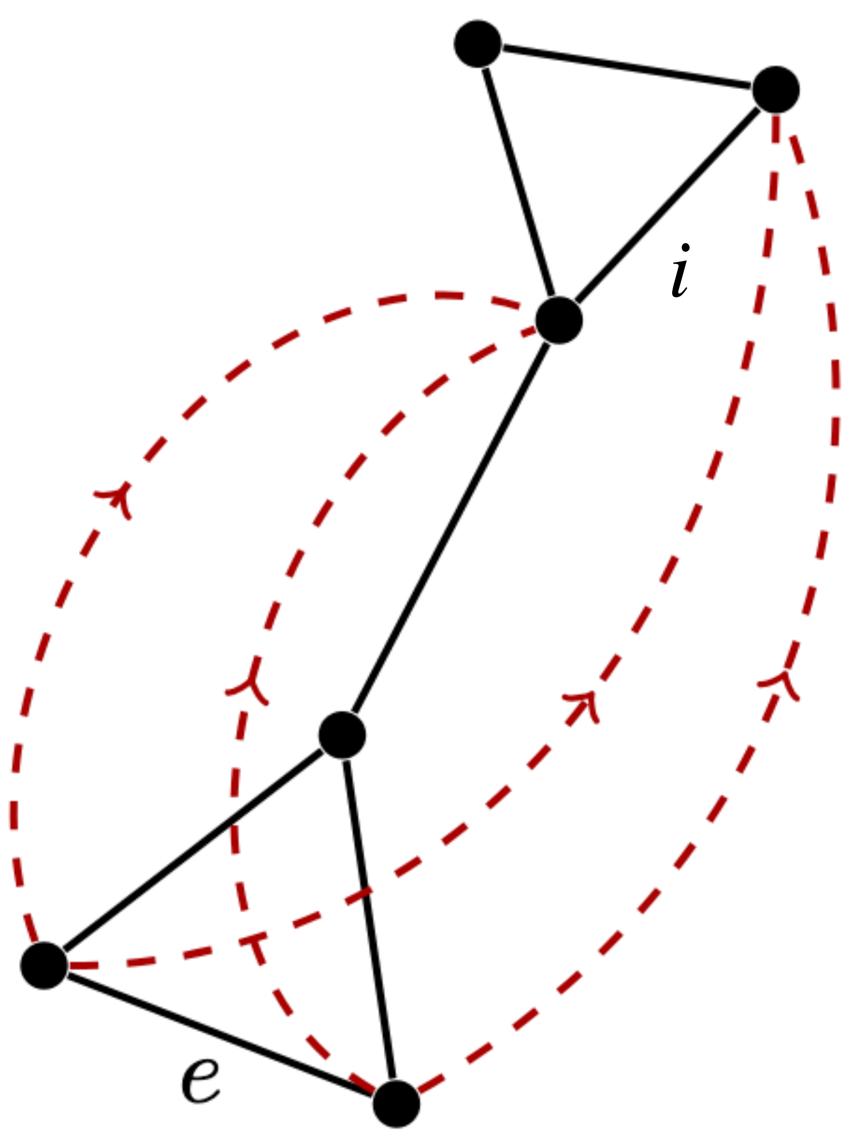
Currents are highly nonlinear in the rates!

Lambda’s in terms of spanning trees?

Spanning tree ensembles

- λ_e^0 is simple! Current from the spanning trees **pruned by the input edge**

$$\boxed{\lambda^0_{\mathrm{xy}} = \frac{r_{\mathrm{xy}}\tau_{\mathrm{y}}^{\backslash i} - r_{\mathrm{yx}}\tau_{\mathrm{x}}^{\backslash i}}{\sum_{\mathrm{z}} \tau_{\mathrm{z}}^{\backslash i}}}$$



- λ_e^1 again a ratio of spanning trees, but in a **different ensemble**…

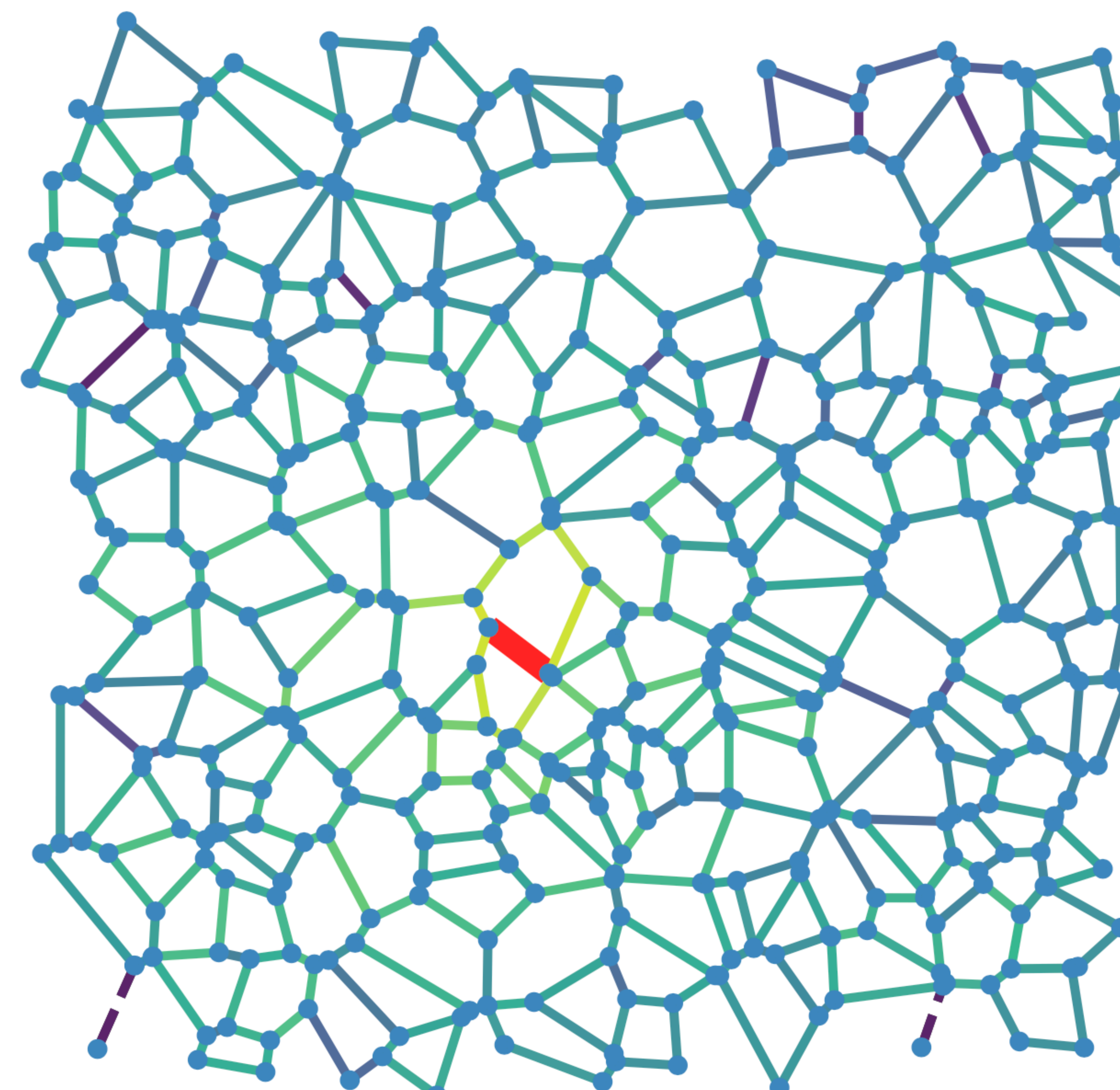
See —> Harunari, Dal Cengio, Lecomte, Polettini, PRL (2024)

More to come…

- A linear relationship arbitrarily far from equilibrium: any connection to **fluctuation theorems**?

- The result applies **beyond Markov chains**

Susceptibility map



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- Does the linearity survive when we control **more than one input current**?

Yes! But you cannot control any arbitrary currents…

(Soon to appear on the ArXiv…)

Thank you for the attention!

- A linear relationship arbitrarily far from equilibrium: any connection to **fluctuation theorems**?
- The result applies **beyond Markov chains**
- Does the linearity survive when we control **more than one input current**?

Yes! But you cannot control any arbitrary currents…

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The End